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1993 J. Phys. A: Math. Gen. 26 4135

(http://iopscience.iop.org/0305-4470/26/16/029)

J. Phys. A: Math. Gen. 26 (1993) 4135–4146. Printed in the UK

On the exact description of the Dirac particle in the scalar and pseudoscalar fields

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Received 1 December 1992

Abstract. The problem of the exact solutions of the Dirac equation in the presence of the scalar and pseudoscalar fields is investigated by means of an algebraic method. Such an approach allows us to find all the configurations of the scalar and pseudoscalar fields where separation of variables is possible. Some general recommendations for the search of exact solutions of the Dirac equation in Cartesian coordinates are proposed and some exact solutions in curvilinear coordinates are discussed.

1. Introduction

It is well known that the description of almost all physical systems can be translated into the language of some universal partial derivative matrix-differential equation of first order [1]. During the past few decades a great number of works have been dedicated to finding methods of exact solutions of such equations. In particular, while looking for a consistent theory of gravitation and quantum theory without contradictions, it is important to have different types of exact solutions of the Dirac equation.

First at all we must note that we have, today, hundreds of exact solutions of the Dirac equation for the case of the external electromagnetic field. A comprehensive review of these solutions is given in the newly published book by Bagrov *et al* [2]. Concerning the Dirac equation in the external gravitational field, here the first publications that must be mentioned are the works of Brill and Wheeler [3] (central symmetrical case) and Chandrasekhar [4, 5] (the Kerr metric, cylindric symmetry). In fact all subsequent publications in these topics continue the investigations of Brill and Wheeler in the case of the diagonal metric and those of Chandrasekhar in the case of the non-diagonal metric [6–26]. Reviews of these publications are given in articles [10, 26].

It is natural that the complete analysis of the Dirac equation may not be limited to the above-mentioned electromagnetic and gravitational fields. Relativistic invariance suggests that the Dirac particle in the general case may be connected with the external world by means of scalar, vector, tensor, pseudovector and pseudoscalar interactions, these being associated with the internal symmetry group of the Dirac equation SU(4). We can also add the possibility of universal gravitational interaction, which as a gauge field is strongly coupled to the Poincaré group. The attractiveness of the general analysis of the Dirac equation taking into account all the types of interactions is confirmed by concrete physical situations, e.g. the constructions of some confinement models (scalar potentials), the interaction of the anomalous magnetic and quadrupole, electric moments of the particle with the tensor of the electromagnetic field, and others.

0305-4470/93/164135+12$07.50 ©1993 IOP Publishing Ltd

The algebraic method of separation of variables proposed by us has been successfully applied to the Dirac equation in different field situations, in particular, in gravitational fields with diagonal [6, 10] and non-diagonal [25, 26] metrics, in an external vector field of a general type (not an obligatory electromagnetic field) [27], and finally, for the most general case, when there are all types of connections, namely scalar, vector, tensor, pseudovector, pseudoscalar and gravitational [9, 28].

As almost all solutions of the Dirac equation are associated with electromagnetic and gravitational fields, the possibilities of exact solutions for the case of external scalar (S) and pseudoscalar (P) fields are considered in this article. A few works are dedicated to this problem. In the best-known articles [29-35], the problem is investigated only for the case of scalar field. The problem of the Dirac particle for different one-dimensional scalar potential has been discussed in [29-32]. In [32] the author shows that the problem can be reduced to the solution of a Riccati equation depending only linearly on the potential. A different approach to this problem, connected with our algebraic method, is to reduce the Dirac equation, after separation of variables, to a system of two first-order coupled ordinary differential equations depending only linearly on the potential [33-35]. Naturally, the problem may be translated to the language of the second-order differential equation for each component of the Dirac bispinor [35].

The most general analysis of the possibilities of exact solutions of the Dirac equation in the scalar and pseudoscalar fields is presented in this article. Two aspects of the problem are investigated here: (1) all scalar and pseudoscalar potentials are found, allowing the complete separation of variables in the Dirac equation, (2) general recommendations for the exact solutions of the Dirac equation in external scalar and pseudoscalar fields are given.

2. Separation of variables

As the algebraic method of separation of variables has been many times described in previous works [6, 9, 10, 27, 28], we limit the discussion here to some main points, and give only finished results of the separation of variables in the Dirac equation in the presence of the scalar and pseudoscalar fields, giving special attention to some peculiar cases.

We write the Dirac equation in the following form

$$\left\{\frac{\gamma^i}{h_i}\partial_i+\frac{\gamma^j}{h_j}\partial_j+\frac{\gamma^m}{h_m}\partial_m+\frac{\gamma^n}{h_n}\partial_n+m_0+U\right\}\Psi=\{H\}\Psi=0. \tag{1}$$

Here h_k are Lamé functions. We have not shown summation on the indices i, j, m, n. Equation (1) is written in the diagonal tetrad gauge, connected with the axes of the corresponding orthogonal curvilinear coordinate system. For the field member we have

$$U=\begin{cases} A_S(x^i,x^j,x^m,x^n) & \text{scalar interaction}\\ \gamma^5 A_P(x^i,x^j,x^m,x^n) & \text{pseudoscalar interaction}\\ A_S+\gamma^5 A_P & \text{coupled S and P interactions.}\end{cases} \tag{2}$$

Redetermining the time variable to be in imaginary units we take all the Dirac matrices to be Hermitian. In this way we do not fix which of the variables are to be time dependent.

We separate the variables according to the scheme [6, 9, 10, 27, 28]

$$\{H\}\Psi \Rightarrow F\{H\}\Gamma\Gamma^{-1}\Psi \Rightarrow (\hat{K}_\alpha + \hat{K}_\beta)\Phi \qquad \Psi = \Gamma\Phi \tag{3}$$

$$(\hat{K}_\alpha + \hat{K}_\beta)\Phi = 0 \tag{4}$$

$$\hat{K}_\alpha \hat{K}_\beta - \hat{K}_\beta \hat{K}_\alpha = 0. \tag{5}$$

Here α and β are collective indices of separating variables. Γ is 4×4 matrix and F is an unknown function meeting the requirement (5) under the most arbitrary field member (2).

The application of scheme (3)–(5) to equation (1) leads to the following results.

Cartesian coordinates

The complete separation of variables in equation (1) in the presence of the scalar or pseudoscalar field is possible only when the field function is dependent on only one variable. In accordance with scheme (3)–(5) we have several ways of separation, all leading to one result, namely, that the field function depends on the variable x^i. i.e. we have

$$\begin{aligned}\hat{K}_i &= \{\gamma^i\partial_i + A_S(x^i) + m_0\}\gamma^j\gamma^m\gamma^n\\ \hat{K}_{jmn} &= \{\gamma^j\partial_j + \gamma^m\partial_m + \gamma^n\partial_n\}\gamma^j\gamma^m\gamma^n\end{aligned} \tag{6}$$

$$\begin{aligned}\hat{K}_i &= \{\gamma^i\partial_i + \gamma^5 A_P(x^i) + m_0\}\gamma^j\gamma^m\gamma^n\\ \hat{K}_{jmn} &= \{\gamma^j\partial_j + \gamma^m\partial_m + \gamma^n\partial_n\}\gamma^j\gamma^m\gamma^n.\end{aligned} \tag{7}$$

If we have scalar and pseudoscalar interactions simultaneously, complete separation of variables in equation (1) is possible when each field function depends on only one variable, but not necessarily the same variable. After applying scheme (3)–(5) we have

$$\begin{aligned}\hat{K}_i &= \{\gamma^i\partial_i + A_S(x^i) + \gamma^5 A_P(x^i) + m_0\}\gamma^j\gamma^m\gamma^n\\ \hat{K}_{jmn} &= \{\gamma^j\partial_j + \gamma^m\partial_m + \gamma^n\partial_n\}\gamma^j\gamma^m\gamma^n\end{aligned} \tag{8}$$

$$\begin{aligned}\hat{K}_{ij} &= \{\gamma^i\partial_i + \gamma^j\partial_j + \gamma^5 A_P(x^i)\}\gamma^m\gamma^n\\ \hat{K}_{mn} &= \{\gamma^m\partial_m + \gamma^n\partial_n + A_S(x^m) + m_0\}\gamma^m\gamma^n.\end{aligned} \tag{9}$$

The separation of variables on which the field functions are not dependent is trivial, and we do not show the corresponding separating operators. The solution of equation (1) for these variables may be written as a free plane wave.

General cylindrical curvilinear coordinates

The new variables in this case are introduced in the plane XY:

$$x = x(\mu, \nu) \qquad y = y(\mu, \nu). \tag{10}$$

Of course, complete separation of variables in equation (1) when the field functions depend on the variables not included in transformation (10), i.e. z and $\tau = it$, is accomplished according to the scheme (3)–(5), but it is necessary to introduce the corresponding Lamé factors before the curvilinear members in the separating operators.

We have a different situation when the field functions depend on the curvilinear variables. Now the complete separation of variables in equation (1) is possible on condition that the Lamé functions depend themselves on only one curvilinear variable, e.g.

$$h_1 = h_2 = \mathscr{F}(\nu). \tag{11}$$

Note that this condition does not consider parabolic cylindrical and elliptic cylindrical coordinates.

The Dirac equation written in the form (1) is connected to the diagonal tetrad gauge [27] and contains only two equal Lamé functions, depending in the general case on two curvilinear coordinates

$$h_1 = h_2 = h(\mu, \nu) \qquad h_3 = h_4 = 1. \tag{12}$$

Complete separation of variables in equation (1) in the case of the presence of one field (scalar or pseudoscalar) as well as in the presence of coupled scalar and pseudoscalar fields is possible when the Lamé functions satisfy the requirement (11) and the field functions depend also only on the variable. After applying scheme (3)-(5) we have the following separating operators

$$\left.\begin{aligned} &\hat{K}_\tau = -\mathrm{i}\partial_\tau \qquad \hat{K}_z = -\mathrm{i}\partial_z \qquad \hat{K}_\mu = -\mathrm{i}\partial_\mu \\ &\hat{K}_\nu = \mathscr{F}(\nu)\left\{\frac{\gamma^\nu}{\mathscr{F}(\nu)}\partial_\nu + A_\mathrm{S}(\nu) + \gamma^5 A_\mathrm{P}(\nu) + \gamma^4\varepsilon + \gamma^z k_z\right\}. \end{aligned}\right. \tag{13}$$

Here ε and k_z are eigenvalues of operators $\hat{K}_\tau$ and $\hat{K}_z$.

If only one field function depends on the variable μ, complete separation of variables in equation (1) is impossible.

Curvilinear coordinates with axis of symmetry

The new space coordinates are connected with the Cartesian coordinates as follows:

$$x = f(\mu, \nu)\cos\varphi \qquad y = f(\mu, \nu)\sin\varphi \qquad z = g(\mu, \nu). \tag{10a}$$

There are now all three Lamé functions in equation (1), and in the case of the diagonal tetrad gauge we have

$$h_\mu = h_\nu = h(\mu, \nu) \qquad h_\varphi = h_\varphi(\mu, \nu). \tag{12a}$$

The complete separation of variables in equation (1), if $A_\mathrm{S} = A_\mathrm{S}(\tau)$ and $A_\mathrm{P} = A_\mathrm{P}(\tau)$, is accomplished according to the scheme (3)-(5) and leads to results analogous to (6)-(9), where the corresponding Lamé factors must be introduced before the curvilinear members. In other cases complete separation of variables is impossible. Even if

$$h_\mu = h_\nu = \mathscr{F}(\nu) \qquad h_\varphi = \mathscr{F}(\nu)\mathscr{G}(\mu) \tag{14}$$

(i.e. general spherical coordinates), and the field functions depend on μ and ν (each or both function on one or both variables), only partial separation of variables is possible in equation (1) (τ and φ from μ and ν). Now the separating operators take the following form

$$\left.\begin{aligned} &\hat{K}_\tau = -\mathrm{i}\partial_\tau \qquad \hat{K}_\varphi = -\mathrm{i}\partial_\varphi \\ &\hat{K}_{\mu\nu} = \mathscr{F}(\nu)\mathscr{G}(\mu)\left\{\frac{\gamma^\mu\partial_\mu}{\mathscr{G}(\nu)} + \frac{\gamma^\nu\partial_\nu}{\mathscr{F}(\nu)} + m_0 + A_\mathrm{S} + \gamma^5 A_\mathrm{P} + \gamma^4\varepsilon + {}^z K_z\right\}. \end{aligned}\right. \tag{15}$$

Note that condition (14) does not consider the coordinates of oblate and prolate ellipsoids (spheroidal coordinates).

So the possibilities of complete separation of variables in the Dirac equation in the presence of scalar and pseudoscalar fields are more limited than in the cases of gravitational [6, 10] and vector [27] fields, in spite of the tensor structure of the scalar and pseudoscalar fields being very simple. The reasons are as follows: The gravitational field is introduced in the Dirac equation through generalized Lamé functions. The requirements of types (11) or (14) lead to some conditions on the gravitational field, but these conditions may be satisfied [6, 10] so long as the geometrical introduction of the gravitational field through generalized Lamé functions using the diagonal tetrad gauge does not lead to the appearance of additional matrices in the Dirac equation. As the electromagnetic (vector) field may be introduced into the Dirac equation by means of the 'lengthening' of the impulse (the minimal inclusion of electromagnetic interaction), this also does not lead to the appearance of new matrices in the Dirac equation. The field configuration doubling in structure by a factor of the corresponding Lamé function (if the requirements (11) or (14) are satisfied) automatically provides complete separation of variables in equation (1) [27]. The inclusion of the scalar or pseudoscalar field introduces into the Dirac equation a new functional dependence with a new matrix factor: this is $IA_S(x^i)$ in the case of the scalar field (compare with the member in the absence of the fields Im_0, where m_0 is the mass of the particle, i.e. a constant value), and $\gamma^5 A_P(x^i)$ in the case of the pseudoscalar field. For the same reasons, the complications of such types takes place under the inclusion of other non-geometrized fields [9, 28].

Special cases of separation of variables

Let us consider the separation of variables in equation (1) in the coordinates excluded by the requirements (11) and (14), using some special similarity transformations [10, 27].

For general cylindrical coordinates, if condition (11) is not satisfied, we must show that for complete separation of variables in the Dirac equation, the interaction is accomplished with only one field (scalar or pseudoscalar), and the field function may depend on only one non-curvilinear variable (z or τ). The scheme of separation is analogous to the situation in the presence of a vector field [27]. Thus, complete separation of variables in the Dirac equation in parabolic cylindrical and elliptic cylindrical coordinates is possible only if the motion is free in the variables μ and ν.

In the case of spheroidal coordinates when the requirement is not satisfied we use the similarity transformation as follows [27]

$$\gamma \to S^{-1}\gamma S \qquad \Phi \to S^{-1}\Psi \qquad SS^{-1} = I$$
$$S = \exp\left(\gamma^4\gamma^3\frac{\theta_1}{2}\right)\exp\left(\gamma^1\gamma^2\gamma^3\gamma^4\frac{\theta_2}{2}\right) \qquad \theta_k = \theta_k(\mu, \nu). \tag{16}$$

The condition

$$\gamma^1 S^{-1}\partial_1 S + \gamma^2 S^{-1}\partial_2 S = 0 \tag{17}$$

is accomplished automatically thanks to the orthogonality of the spheroidal coordinates.

Now in order to separate the variables in equation (1) completely we must have

$$h_1^2 = h_2^2 = a^2 - b^2 = c^2 - d^2 \qquad h_3 = ab. \tag{18}$$

After transformation (16), taking into account the relation (17), we have that the complete separation of variables in equation (1) is possible only if

$$\begin{aligned} a &= a(\mu) \text{ or } a(\nu) \qquad & c &= c(\mu) \text{ or } c(\nu) \\ b &= b(\mu) \text{ or } b(\nu) \qquad & d &= d(\mu) \text{ or } d(\nu). \end{aligned} \tag{19}$$

Each function a, b, c, d depends on only one (not necessarily the same) curvilinear variable μ or ν.

Equation (1) now takes the form

$$\begin{aligned} \Big\{\gamma^1\partial_1 + \gamma^2\partial_2 + \left(\frac{1}{b} + \frac{1}{a}\gamma^4\gamma^3\right)\gamma^3\partial_3 + (a + b\gamma^4\gamma^3)\gamma^4\partial_4 \\ + (m_0 + A_S + \gamma^5 A_P)(c + d\gamma^1\gamma^2\gamma^3\gamma^4)\Big\}\Phi = 0 \end{aligned} \tag{20}$$

$$\mu = x^1, \qquad \nu = x^2 \qquad \varphi = x^3 \qquad \tau = x^4.$$

Taking into account the scheme (3)-(5) applied to equation (20) we have that complete separation of variables is provided by the following configurations of scalar and pseudoscalar potentials

$$A_S = \frac{c\xi + \mathrm{d}\zeta}{c^2 + \mathrm{d}^2} - m_0 \qquad A_P = \frac{c\zeta - \mathrm{d}\xi}{c^2 + \mathrm{d}^2} \tag{21}$$

where ξ and ζ are arbitrary functions of one of the variables μ or ν, each depending on its own variables.

So the very complicated coordinates of prolate and oblate ellipsoids allow the complete separation of variables in the Dirac equation in the presence of the field (21).

3. Exact solutions

General recommendations

In view of the simplicity of the separating operators connected with scalar and pseudoscalar fields, it is possible to propose some general recommendations on the exact solutions of equation (1) even without concretization of field functions in its dependence on coordinate variables.

Passing to the problem on the eigenvalues we have from (6)

$$\hat{K}_i\Phi = -\hat{K}_{jmn}\Phi = -k\Phi. \tag{22}$$

Now we accept

$$x^i = x \qquad x^j = y \qquad x^m = z \qquad x^n = \mathrm{i}t. \tag{23}$$

we find the solution in the form

$$\Phi = \Theta \exp\{-\mathrm{i}(\varepsilon t - k_y y - k_z z)\} \qquad k^2 = \varepsilon^2 - k_y^2 - k_z^2. \tag{24}$$

It is natural that in the absence of field we have

$$k^2 = k_x^2 + m_0^2 \rightarrow \varepsilon^2 = k_x^2 + k_y^2 + k_z^2 + m_0^2. \tag{25}$$

The function Θ, according to (6), is defined by the equation

$$\{\gamma^5\partial_x + \gamma^1\gamma^5[A_S(x) + m_0] + k\}\Theta = 0. \tag{26}$$

In the representation of the Dirac matrices in the form

$$\gamma^1=\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix} \qquad \gamma^2=\mathrm{i}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$$
$$\gamma^3=\begin{pmatrix}-\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix} \qquad \gamma^4=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix} \tag{27}$$

for the components of the bi-spinor Θ we have

$$\Theta_1(\Theta_4)=\exp\left\{\int [A_S(x)+m_0]\,\mathrm{d}x\right\}\tilde\Theta_1(\tilde\Theta_4) \qquad \tilde\Theta_4=c_1\tilde\Theta_1$$
$$\Theta_2(\Theta_3)=\exp\left\{-\int [A_S(x)+m_0]\,\mathrm{d}x\right\}\tilde\Theta_2(\tilde\Theta_3) \qquad \tilde\Theta_3=c_2\tilde\Theta_2 \tag{28}$$

where $\tilde\Theta_k$ are defined by means of the ordinary differential equations of second order

$$\frac{\mathrm{d}^2\tilde\Theta_1}{\mathrm{d}x^2}+2[A_S(x)+m_0]\frac{\mathrm{d}\tilde\Theta_1}{\mathrm{d}x}+k^2\tilde\Theta_1=0$$
$$\frac{\mathrm{d}^2\tilde\Theta_2}{\mathrm{d}x^2}-2[A_S(x)+m_0]\frac{\mathrm{d}\tilde\Theta_2}{\mathrm{d}x^2}+k^2\tilde\Theta_2=0. \tag{29}$$

The solution of these equations is possible only after concretization of the function A_S.

In the case of the pseudoscalar field, the relations (22)-(24) also take place, but the bi-spinor Θ, in accordance with (7), is described by the equation

$$\{\gamma^5\partial_x+\gamma^1A_P(x)+\gamma^1\gamma^5m_0+k\}\Theta=0. \tag{30}$$

Accepting the representation

$$\gamma^1=\begin{pmatrix}\sigma_1 & 0\\ 0 & -\sigma_1\end{pmatrix} \qquad \gamma^2=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$
$$\gamma^3=\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix} \qquad \gamma^4=\mathrm{i}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} \tag{31}$$

and introducing a new unknown function according to

$$\Theta_1(\Theta_3)=\exp\left\{-\int A_P(x)\,\mathrm{d}x\right\}\tilde\Theta_1(\tilde\Theta_3) \qquad \tilde\Theta_3=c_1\tilde\Theta_1$$
$$\Theta_2(\Theta_4)=\exp\left\{\int A_P(x)\,\mathrm{d}x\right\}\tilde\Theta_2(\tilde\Theta_4) \qquad \tilde\Theta_4=c_2\tilde\Theta_2 \tag{32}$$

for the components $\tilde\Theta_k$, we have

$$\frac{\mathrm{d}^2\tilde\Theta_1}{\mathrm{d}x^2}+2A_P(x)\frac{\mathrm{d}\tilde\Theta_1}{\mathrm{d}x}-(m_0^2-k^2)\tilde\Theta_1=0$$
$$\frac{\mathrm{d}^2\tilde\Theta_2}{\mathrm{d}x^2}-2A_P(x)\frac{\mathrm{d}\tilde\Theta_2}{\mathrm{d}x}-(m_0^2-k^2)\tilde\Theta_2=0. \tag{33}$$

Unfortunately, we may not propose such simple recommendations in the case (8) when Θ is defined by the equation

$$\{\gamma^5\partial_x+\gamma^1\gamma^5[A_S(x)+m_0]+\gamma^1A_P(x)+k\}\Theta=0 \tag{34}$$

because it is not possible to reduce all the matrices of the equation γ^1, γ^5 and $\gamma^1\gamma^5$ to one and the same structure.

Regarding separation according to scheme (9), here the simplifications analagous to (28), (29), (32), (33) are possible, however, the optimal representation of the Dirac matrices must be accepted for each of the operators (9). This fact must be taken into account subsequently in the reconstruction of the bi-spinor.

In particular we have, from (9)

$$\{\gamma^1\partial_x+\gamma^5 A_P(x)+\mathrm{i}\gamma^2 k_y+k\}\gamma^3\gamma^4\Phi=0 \tag{35}$$

$$\{\gamma^3\partial_z+[A_S(x)+m_0]-\mathrm{i}\gamma^4\varepsilon-k\}\gamma^3\gamma^4\Phi=0. \tag{36}$$

Accepting for (35) the representation

$$\begin{aligned}\gamma^1&=\begin{pmatrix}\sigma_1 & 0\\ 0 & -\sigma_1\end{pmatrix} \qquad \gamma^2=\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix}\\ \gamma^3&=\mathrm{i}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} \qquad \gamma^4=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\end{aligned} \tag{37}$$

we have

$$\begin{aligned}\gamma^3\gamma^4\Phi_1(\Phi_3)&=\exp\int A_P(x)\,\mathrm{d}x\;\tilde{\Phi}_1(\tilde{\Phi}_3) \qquad \tilde{\Phi}_3=c_1\tilde{\Phi}_1\\ \gamma^3\gamma^4\Phi_2(\Phi_4)&=\exp\left\{-\int A_P(x)\,\mathrm{d}x\right\}\tilde{\Phi}_2(\tilde{\Phi}_4) \qquad \Phi_4=c_2\Phi_2\end{aligned} \tag{38}$$

and

$$\begin{aligned}&\frac{\mathrm{d}^2\tilde{\Phi}_1}{\mathrm{d}x^2}-2A_P(x)\frac{\mathrm{d}\tilde{\Phi}_1}{\mathrm{d}x}+(k^2+k_y^2)\tilde{\Phi}_1=0\\ &\frac{\mathrm{d}^2\tilde{\Phi}_2}{\mathrm{d}x^2}+2A_P(x)\frac{\mathrm{d}\tilde{\Phi}_2}{\mathrm{d}x}+(k^2+k_j^2)\tilde{\Phi}_2=0.\end{aligned} \tag{39}$$

Analogously we have from (36)

$$\begin{aligned}\gamma^1&=\begin{pmatrix}\sigma_1 & 0\\ 0 & -\sigma_1\end{pmatrix} \qquad \gamma^2=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\\ \gamma^3&=\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} \qquad \gamma^4=\begin{pmatrix}\sigma_2 & 0\\ 0 & -\sigma_2\end{pmatrix}\end{aligned} \tag{40}$$

$$\begin{aligned}\gamma^3\gamma^4\Phi_1(\Phi_3)&=\exp\left\{-\int[A_S(z)+m_0-k]\,\mathrm{d}z\right\}\tilde{\Phi}_1(\tilde{\Phi}_3)\\ \gamma^3\gamma^4\Phi_2(\Phi_4)&=\exp\int[A_S(z)+m_0-k]\,\mathrm{d}z\;\tilde{\Phi}_2(\tilde{\Phi}_4)\end{aligned} \tag{41}$$

$$\tilde{\Phi}_3=c_1\tilde{\Phi}_1 \qquad \tilde{\Phi}_4=c_2\tilde{\Phi}_2$$

$$\begin{aligned}&\frac{\mathrm{d}^2\tilde{\Phi}_1}{\mathrm{d}x^2}+2[A_S(z)+m_0-k]\frac{\mathrm{d}\tilde{\Phi}_1}{\mathrm{d}x}-\varepsilon^2\tilde{\Phi}_1=0\\ &\frac{\mathrm{d}^2\tilde{\Phi}_2}{\mathrm{d}x^2}-2[A_S(z)+m_0-k]\frac{\mathrm{d}\tilde{\Phi}_2}{\mathrm{d}x}-\varepsilon^2\tilde{\Phi}_2=0.\end{aligned} \tag{42}$$

The concrete sets of A_S and A_P allowing the exact solutions of the equations (29), (33), (39), (42) in the special functions are given in [36, 37].

Concrete examples

As we cannot propose such simple recommendations for the exact solutions of the Dirac equation in the case of curvilinear coordinates as has been proposed in the case of Cartesian coordinates in the previous subsection, we are limited here to some concrete examples.

(a) Let us consider the interaction of the Dirac particle with the scalar field $S=s/r$. The Dirac equation now takes the form

$$\left\{\gamma^1\partial_r+\frac{1}{3}\left(\gamma^2\nabla_\theta+\frac{\gamma^3\partial_\varphi}{\sin\theta}\right)+\mathrm{i}\gamma^4\partial_t+m_0+\frac{s}{r}\right\}\Psi=0. \tag{43}$$

Taking into account the scheme of separation (3) and believing $\Gamma=\gamma^1\gamma^4$ we have

$$\left\{\gamma^4\partial_r+\frac{1}{r}\left(\gamma^2\gamma^1\gamma^4\partial_\theta+\frac{\gamma^3\gamma^1\gamma^4}{\sin\theta}\partial_\varphi\right)-\mathrm{i}\gamma^1\partial_t+\left(m_0+\frac{s}{r}\right)\gamma^1\gamma^4\right\}\Phi=\{H\}\Phi=0. \tag{44}$$

As the angular operator

$$\hat{K}=\gamma^2\gamma^1\gamma^4\partial_\theta+\frac{\gamma^3\gamma^1\gamma^4}{\sin\theta}\partial_\varphi \tag{45}$$

is commuted with $\{H\}$, equation (44) may be rewritten as follows:

$$\left\{\gamma^4\partial_r+\frac{k}{r}-\varepsilon\gamma^1+\left(m_0+\frac{s}{r}\right)\gamma^1\gamma^4\right\}\Phi=0 \tag{46}$$

where k is the constant of separation, i.e. the eigenvalue of the operator k.

Note that equation (43) is written in the diagonal tetrad gauge and therefore the operator $\hat{K}$ is different from the analogous operator written in the Cartesian tetrad gauge. But as has been shown by Brill and Wheeler [3], the radial equation in our situation must not depend on the tetrad gauge. As the angle varies here the so-called 'coordinate effect' takes place.

Equation (46) in the representation

$$\begin{aligned}&\gamma^1=\begin{pmatrix}\sigma^1&0\\0&-\sigma^1\end{pmatrix}\qquad\gamma^4=\begin{pmatrix}\sigma^3&0\\0&-\sigma^3\end{pmatrix}\\&\gamma^1\gamma^4=-\mathrm{i}\begin{pmatrix}\sigma_1&0\\0&\sigma_1\end{pmatrix}\qquad\Phi=\begin{pmatrix}\varphi\\\chi\end{pmatrix}\end{aligned} \tag{47}$$

may be reduced to a system

$$\left\{\sigma^3\partial_r+\frac{k}{r}-\varepsilon\sigma^1-\mathrm{i}\left(m_0+\frac{s}{r}\right)\sigma^2\right\}\varphi=0 \tag{48}$$

$$\left\{-\sigma^3\partial_r+\frac{k}{r}+\varepsilon\sigma^1-\mathrm{i}\left(m_0+\frac{s}{r}\right)\sigma^2\right\}\chi=0. \tag{49}$$

Finally equation (48) in the standard representation of the Pauli matrices

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad\sigma^2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}\qquad\sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{50}$$

leads to the equations

$$\begin{aligned}
&\left(\partial_r+\frac{k}{r}\right)\varphi_1-\left(m_0+\varepsilon+\frac{s}{r}\right)\varphi_2=0\\
&\left(-\partial_r+\frac{k}{r}\right)\varphi_2+\left(m_0-\varepsilon+\frac{s}{r}\right)\varphi_1=0
\end{aligned}\tag{51}$$

which coincide with the well known radial equations of the hydrogen atom. They have exact solutions in terms of the degenerate hypergeometric functions.

So we have standard results here: the scalar potential may lead to confinement.

(*b*) The interaction of the Dirac particle with the pseudoscalar field is described by the equation

$$\left\{\gamma^1\partial_r+\frac{1}{r}\left(\gamma^2\partial_\theta+\frac{\gamma^3\partial_\varphi}{\sin\theta}\right)+\mathrm{i}\gamma^4\partial_t+m_0+P\gamma^5\right\}\Psi=0.\tag{52}$$

If we accept $P=p/r$, after separation of variables we have

$$\{H\}\Phi=\left\{\gamma^4\partial_r+\frac{k}{r}-\varepsilon\gamma^1+m_0\gamma^1\gamma^4\right\}\Phi=0\qquad\Psi=\gamma^1\gamma^4\Phi\tag{53}$$

$$\hat{K}=\gamma^2\gamma^1\gamma^4\partial_\theta+\frac{\gamma^3\gamma^1\gamma^4}{\sin\theta}\partial_\varphi-p\tag{54}$$

$$\hat{K}\{H\}-\{H\}\hat{K}=0.\tag{55}$$

This result may be treated as follows: the particle is 'free' in the radial variable, but its moment is renormalized by the pseudoscalar field.

(*c*) A very interesting situation occurs in the case of the zero mass rest particle interacting with the scalar field $S=s/r$ and pseudoscalar field $P=p/r$ simultaneously. The Dirac equation now may be written as

$$\left\{\gamma^1\partial_r+\frac{1}{r}\left(\gamma^2\partial_\theta+\frac{\gamma^3\partial_\varphi}{\sin\theta}\right)+\mathrm{i}\gamma^4\partial_t+m_0+\frac{s}{r}+\frac{p}{r}\gamma^5\right\}\Psi=0.\tag{56}$$

Separation of variables in this equation according to scheme (3) may be realized in two ways:

(i) $$\Gamma=\gamma^1\gamma^4\tag{57}$$

$$\left\{\gamma^4\partial_r+\frac{1}{r}\left(\gamma^2\gamma^1\gamma^4\partial_\theta+\frac{\gamma^3\gamma^1\gamma^4}{\sin\theta}\partial_\varphi-\gamma^2\gamma^2p\right)-\mathrm{i}\gamma^1\partial_t+\gamma^1\gamma^4\frac{s}{r}\right\}\Phi=0\tag{58}$$

(ii) $$\Gamma=\gamma^2\gamma^3\tag{59}$$

$$\left\{\gamma^1\gamma^2\gamma^3\partial_r+\frac{1}{r}\left(\gamma^3\partial_\theta-\frac{\gamma^2\partial_\varphi}{\sin\theta}+\gamma^2\gamma^3s\right)+\mathrm{i}\gamma^4\gamma^2\gamma^3\partial_t-\gamma^1\gamma^4\frac{p}{r}\right\}\Phi=0.\tag{60}$$

As the commuting relations between matrices before identical members of the equations (58) and (60) coincide, the same equations also coincide up to change $S\rightleftarrows P$. Returning to the previous examples we note that the particle with non-zero rest mass may distinguish between the scalar and pseudoscalar fields, and the zero rest mass particle cannot distinguish between the two fields.

Both equations (58) and (60) lead to the same radial equations that coincide with equations (51). The corresponding exact solutions are well known.

4. Discussion

Concluding the present article we note that all the possible configurations of scalar and pseudoscalar fields allowing complete separation of variables in the Dirac equation for the Cartesian and curvilinear coordinates have been found here. As the exact solutions in the enumerated field situations we have here the necessary systems of second-order differential equations depending only linearly on the S and P potentials. This result coincides with the partial results of other authors [29–35]. Some concrete differences between our results and the results of [29–35] are connected with the fact that we have considered the proper scalar potential when the field function is introduced into the equation with unit matrix factor. [35] considers the problem of the fourth component of the four-potential and the field function is introduced into the equation with the matrix factor γ^4 (see also the work by Cook [38]).

Some words may be said on the unlooked-for 'symmetry' $S \rightleftarrows P$ for the zero rest mass particle although this fact is outside the limits of this article. Taking into account the Dirac equation for the zero rest mass particle interacting with the scalar and pseudoscalar fields

$$\{\gamma^k \partial_k + S + \gamma^5 P\}\Psi = 0 \tag{61}$$

and multiplying it by the matrix γ^5 on the left, we have

$$\{\gamma^5\gamma^k \partial_k + \gamma^5 S + \gamma^5\gamma^5 P\}\Psi = 0. \tag{62}$$

Accepting the representation

$$\begin{aligned} &(\gamma^1)^+ = \gamma^1 \qquad (\gamma^2)^+ = \gamma^2 \qquad (\gamma^3)^+ = \gamma^3 \\ &(\gamma^4)^+ = -\gamma^4 \qquad (\gamma^5)^+ = -\gamma^5 \end{aligned} \tag{63}$$

we have

$$\begin{aligned} &[\gamma^k, \gamma^l]^+ = [\gamma^5\gamma^k, \gamma^5\gamma^l]_+ = 2\delta_{kl} I \\ &\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \gamma^5\gamma^1\gamma^5\gamma^2\gamma^5\gamma^3\gamma^5\gamma^4 \qquad (\gamma^5)^2 = -I \end{aligned} \tag{64}$$

i.e. γ^k and $\gamma^5\gamma^k$ are connected by means of some similarity transformation

$$S\gamma^k S^{-1} = \gamma^5\gamma^k. \tag{65}$$

As a result the equations (61) and (62) may be rewritten in the following form

$$\{\gamma^5\gamma^k \partial_k + S + \gamma^5 P\}\Psi' = 0 \tag{61a}$$

$$\{\gamma^k \partial_k + \gamma^5 S + P\}\Psi' = 0 \qquad \Psi' = S\Psi. \tag{62a}$$

Both the equations (61), (62) and (61a), (62a) lead to the following form of the Dirac equation:

$$\{(1 \pm \gamma^5)\gamma^k \partial_k + (1 \pm \gamma^5)S \mp (1 \mp \gamma^5)P\}\Psi = 0 \tag{66}$$

and the Fermi current takes the form

$$J = \Psi^+(1 \pm \gamma^5)\gamma^k\Psi. \tag{67}$$

So the procedure of the separation of variables in the Dirac equation for the massless particle automatically allows the possibility of two different helicities of the neutrino. The analogous result takes place for the massless Dirac particle in the external gravitational field as has been shown in [10].

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